

An Elementary Integrality Proof of Rothblum's Stable Matching Formulation

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Abstract

In this paper we provide a short new proof for the integrality of Rothblum's linear description of the convex hull of incidence vectors of stable matchings in bipartite graphs. The key feature of our proof is to show that extreme points of the formulation must have a 0, 1-component.

Keywords: Stable Matching, Polytope, Extreme Points

1. Introduction

In an instance of the *stable marriage* problem, we are given a bipartite graph $G = (\mathcal{A} \cup \mathcal{B}, E)$ where \mathcal{A} and \mathcal{B} traditionally represent sets of women and men, respectively. An edge $ab \in E$ corresponds to an *acceptable* pair a and b of man and woman. In the following, we let $N(u) := \{v : uv \in E\}$ be the set of neighbours of u in G . Each node $u \in V := \mathcal{A} \cup \mathcal{B}$ specifies a complete *preference order* $>_u$ over its neighbours where node u prefers neighbour v_1 over v_2 iff $v_1 >_u v_2$. For ease of notation, we will think of $>_u$ as a total ordering on $N(u) \cup \{\emptyset\}$ where \emptyset is the least preferred element of each node $u \in V$. In their seminal paper [GS62], Gale and Shapley introduced the above problem, and provided a constructive proof of existence of so called *stable* matchings. A matching is a collection M of edges in E such that each node is incident to at most one edge in M . M is stable if, for every edge $uv \notin M$, $M(u) >_u v$ or $M(v) >_v u$ where $M(u)$ is the node matched to u in M if that exists, and $M(u) := \emptyset$, otherwise.

In this paper, we focus on polyhedral characterizations of the set of incidence vectors of stable matchings. Vande Vate first provided such a description in [Vat89] for the special case where G is a complete bipartite graph. Rothblum [Rot92] later generalized Vande Vate's result to incomplete preference lists and simplified the proof of integrality.

We provide an even simpler, more compact argument for the integrality of Rothblum's formulation. Our arguments are elementary and rely solely on some well-known results on the symmetric difference of stable matchings as well as some knowledge of the local structure of extreme points in our formulation to achieve the desired result.

Necessary background from the literature will be covered in the section to follow. The main result, a one-page proof, will be given in section 3.

2. Stable Matchings Preliminaries

We briefly review a couple of well-known facts on stable matchings. For each edge uv in E , we let $\delta^{>u}(v) := \{\{v, w\} \in E : w >_v u\}$ to be the set of edges incident to v and those of its neighbours that are preferred to u . For $v \in V$ let $N_{\max}(v)$ denote its most preferred neighbour. A matching M in G is now easily seen to be stable if

$$M \cap (\delta^{>u}(v) \cup \delta^{>v}(u) \cup \{e\}) \neq \emptyset, \quad (1)$$

for all $e = uv \in E$. The following lemmas study the connected components of the *symmetric difference* $M_1 \Delta M_2 := (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ of stable matchings M_1 and M_2 . Note that $M_1 \Delta M_2$ is an edge set, however in this paper we refer to nontrivial connected components of the graph $(V, M_1 \Delta M_2)$ as connected components of $M_1 \Delta M_2$. Here, $V(C)$ and $E(C)$ denote the set of nodes and edges, respectively, of a graph C .

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Lemma 2.1 Let M_1 and M_2 be two stable matchings in G and let J be a connected component in $M_1 \Delta M_2$. Then for some $\{i, j\} = \{1, 2\}$, we have

$$M_i(a) >_a M_j(a) \text{ and } M_j(b) >_b M_i(b), \quad (2)$$

for all $a \in V(J) \cap \mathcal{A}$ and $b \in V(J) \cap \mathcal{B}$.

Proof: Since M_1 and M_2 are matchings, J is a path or a cycle with edges alternating between M_1 and M_2 . Let $v \in V$ be an end node of J if J is a path, otherwise let v be an arbitrary node of J . For visualization of the proof see Figure 1.

W.l.o.g. $a := v$, $a \in \mathcal{A}$ and $b := M_1(a) >_a M_2(a)$. If $a = M_1(b) >_b M_2(b)$, the matching M_2 violates (1) for the edge $ab \in E$. Thus, $M_2(b) >_b M_1(b) = a$. Thus, $M_2(b) \neq \emptyset$ and the matching M_1 satisfies (1) for the edge $e := bM_2(b) \in E$ only if $M_1(M_2(b)) >_{M_2(b)} b$. Continuing in this way, we obtain statement (2). ■

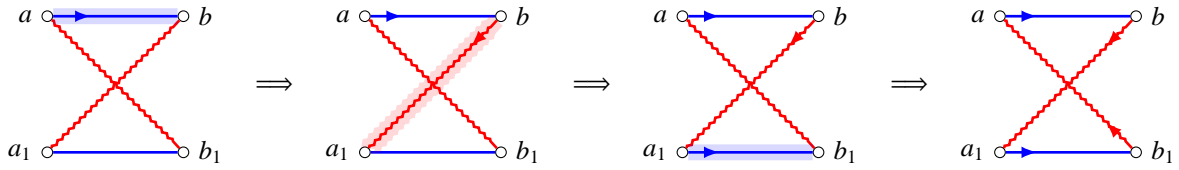


Figure 1: Visualizing the proof of Lemma 2.1. Here, the connected component J is a cycle on four nodes. The edges of M_1 are marked by straight blue lines, and the edges of M_2 by red zigzags. So here $a_1 = M_2(b)$ and $b_1 = M_1(M_2(b))$. For each node $w \in \{a, b, a_1, b_1\}$, the arrow at w points towards the most preferred node in $\{M_1(w), M_2(w)\}$ with respect to $>_w$.

The next Lemma is equivalent to Theorem 2.16 in [RS92]. In the interest of self-containment we provide a short elementary proof below.

Lemma 2.2 Let M_1 and M_2 be two stable matchings in G . Let J_1 be those connected components of $M_1 \Delta M_2$ that satisfy (2) for $i = 1$ and $j = 2$ (i.e., \mathcal{A} nodes prefer M_1 edges); let J_2 be all remaining connected components of $M_1 \Delta M_2$. Then both $M'_1 = M_1 \Delta E(J_1)$ and $M'_2 = M_1 \Delta E(J_2)$ are stable matchings in G .

Proof: For contradiction assume that one of the matchings M'_1 and M'_2 is not stable; w.l.o.g. assume that M'_1 does not satisfy (1) for some edge $ab \in E$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. For visualization of the proof see Figure 2.

Since M_1 and M_2 are stable, a and b lie in $V(J_1) \cup V(J_2)$. Otherwise $M'_1(a) = M_1(a)$, $M'_1(b) = M_1(b)$ or $M'_1(a) = M_2(a)$, $M'_1(b) = M_2(b)$, and thus one of M_1, M_2 also violates (1) for edge ab . Similarly, a and b cannot both lie in $V(J_1)$ or both in $V(J_2)$. Suppose first that $a \in V(J_1)$ and $b \in V(J_2)$. In this case $M'_1(a) = M_2(a)$ and $M'_1(b) = M_1(b) >_b M_2(b)$. Thus M_2 violates (1) for edge ab .

If $a \in V(J_2)$ and $b \in V(J_1)$, then $M'_1(a) = M_1(a)$ and $M'_1(b) >_b M_1(b)$, and hence M_1 violates (1) for edge ab , contradiction. ■

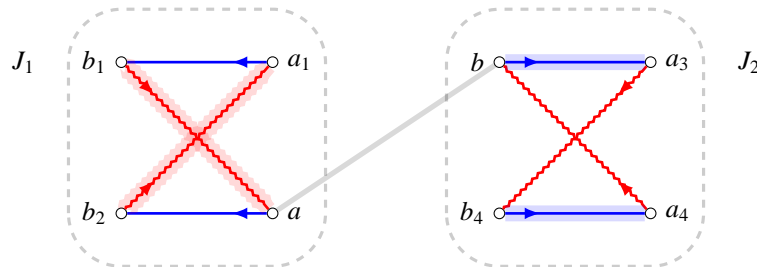


Figure 2: Visualizing the proof of Lemma 2.2. Here, both J_1 and J_2 consist of one cycle on four nodes. The edges of M_1 are marked by blue straight lines, and the edges of M_2 by red zigzags; the edges of $M_1 \Delta E(J_1)$ are the highlighted edges of M_1 and M_2 . For each node w , the arrow at w points towards the most preferred node in $\{M_1(w), M_2(w)\}$ with respect to $>_w$. The figure illustrates the case, when the matching $M'_1 = M_1 \Delta E(J_1)$ violates (1) for the edge $\{a, b\}$ with $a \in V(J_1)$ and $b \in V(J_2)$. So here $M'_1(a) = M_2(a) = b_1$ and $M'_1(b) = M_1(b) = a_3 >_b a_4 = M_2(b)$. In this case, M_2 violates (1) for the edge $\{a, b\}$ as well, contradiction.

Definition 2.3 Let us define the stable matching polytope $P(G) \subseteq \mathbb{R}^E$ for graph G as follows

$$P(G) := \text{conv}\{\chi(M) \in \mathbb{R}^E : M \text{ is a stable matching in } G\}.$$

By [GS62], $P(G)$ is a nonempty polytope because every graph G has a stable matching.

Clearly, the vertices of $P(G)$ are in one-to-one correspondence with stable matchings in G . Moreover, Lemma 2.2 helps to understand what pairs of stable matchings in G do not correspond to edges of $P(G)$.

Lemma 2.4 Let M_1 and M_2 be two stable matchings in G which define an edge of the polytope $P(G)$. Then all connected components in $M_1 \Delta M_2$ satisfy (2) for unique choice of i and j .

Proof: Suppose for contradiction that the statement of the lemma does not hold. Hence the sets J_1 and J_2 are both nonempty in Lemma 2.2, and we obtain stable matchings $M_1 \Delta E(J_1)$ and $M_1 \Delta E(J_2)$ that are different from M_1, M_2 . We also have

$$\frac{1}{2}\chi(M_1 \Delta E(J_1)) + \frac{1}{2}\chi(M_1 \Delta E(J_2)) = \frac{1}{2}\chi(M_1) + \frac{1}{2}\chi(M_2),$$

and hence there are two distinct convex combinations of the midpoint of the edge between M_1 and M_2 ; a contradiction. ■

The next Corollary can be obtained from Ratier's characterization of edges of the stable matching polytope [Rat96].

Corollary 2.5 Let M_1 and M_2 be two stable matchings in G such that

$$M_1 \cap \delta^{>a}(b) \neq \emptyset, M_1 \cap \delta^{>b}(a) \neq \emptyset \quad \text{and} \quad M_2 \cap (\delta^{>a}(b) \cup \delta^{>b}(a)) = \emptyset \quad (3)$$

for some $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then, M_1 and M_2 do not define an edge of the polytope $P(G)$.

Proof: Condition (3) implies that both a and b prefer M_1 over M_2 . Hence, both sets J_1 and J_2 as given in Lemma 2.2 must be non-empty. An application of Lemma 2.4 completes the proof of the corollary. ■

3. Linear Description

Let us define $Q(G) \subseteq \mathbb{R}^E$ to be the polytope described by the following linear constraints

$$x(\delta(v)) \leq 1, \quad \forall v \in V \quad \text{and} \quad x_e \geq 0, \quad \forall e \in E, \quad (4)$$

$$x(\delta^{>a}(b)) + x(\delta^{>b}(a)) + x_{ab} \geq 1, \quad \forall ab \in E \quad (5)$$

where $x(J) := \sum_{e \in J} x_e$ for any $J \subseteq E$.

Clearly, $P(G) \subseteq Q(G)$ because for every stable matching M in G the point $x := \chi(M)$ satisfies (4) and by (1) the point x also satisfies (5). On the other hand, every integral point in $Q(G)$ equals $\chi(M)$ for some stable matching M in G . In the remaining part of the paper we show that every vertex of $Q(G)$ is integral, thus proving the main theorem.

Theorem 3.1 For every graph G the polytope $P(G)$ equals $Q(G)$.

Lemma 3.2 For every graph G every vertex of the polytope $Q(G)$ is integral.

Proof: We first claim that every vertex x of $Q(G)$ satisfies $x_e \in \{0, 1\}$ for at least one $e \in E$. Assume for contradiction that $0 < x_e < 1$ for all $e \in E$. Like every vertex of $Q(G)$, x is uniquely defined by $|E|$ linearly independent tight constraints describing $Q(G)$. Since x has no zero coordinate, we can assume that these constraints are the constraints $x(\delta(v)) \leq 1$ for $v \in V_x$ and the constraints (5) for $e \in E_x$, where $|V_x| + |E_x| = |E|$. Moreover, let us assume that we choose the $|E|$ tight constraints so that $|V_x|$ is as large as possible.

The constraints $x(\delta(v)) = 1, v \in V$ are linearly dependent, since $\sum_{a \in \mathcal{A}} x(\delta(a)) = \sum_{b \in \mathcal{B}} x(\delta(b))$ for every $x \in \mathbb{R}^E$. Hence, we have $V_x \subsetneq V$. On the other hand if $a = N_{\max}(b)$ for some $b \in V$ then $e := ab \notin E_x$. Indeed, $a = N_{\max}(b)$ implies $\delta^{>a}(b) = \emptyset$, then

$$1 \leq x(\delta^{>a}(b)) + x(\delta^{>b}(a)) + x_{ab} = x(\delta^{>b}(a)) \leq x(\delta(a)) \leq 1,$$

showing that $\delta^{<b}(a) = \emptyset$. So by linear independence we cannot have both $e \in E_x$ and $a \in V_x$. Suppose for a contradiction $e \in E_x$. Then $a \notin V_x$ and hence $|E_x \setminus \{e\}| + |V_x \cup \{a\}| = |E|$, and $|V_x \cup \{a\}| > |V_x|$. Moreover $E_x \setminus \{e\}$ and $V_x \cup \{a\}$ also define the vertex x . This contradicts the choice of V_x, E_x . Analogously, we can show that if $b \in V_x$ and $a = N_{\min}(b)$ then $e := ab \notin E_x$. Moreover, notice that $N_{\min}(v) \neq N_{\max}(v)$ for $v \in V_x$ since no coordinate of x equals 1. Thus,

$$|E_x| = \frac{1}{2} \sum_{v \in V} |\delta(v) \cap E_x| \leq \frac{1}{2} \sum_{v \in V_x} (|\delta(v)| - 2) + \frac{1}{2} \sum_{v \in V \setminus V_x} (|\delta(v)| - 1) = |E| - |V_x| - \frac{1}{2}|V \setminus V_x|,$$

which implies $|E_x| + |V_x| < |E|$, contradiction.

Now let us assume that G is a graph with the minimum number of edges such that $Q(G)$ is not an integral polytope. Let x be a non-integral vertex of $Q(G)$.

Case $x_{ab} = 0$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $e := ab \in E$. In this case, let P' and x' be obtained from $Q(G) \cap \{x \in \mathbb{R}^E : x_{ab} = 0\}$ and x by dropping the coordinate corresponding to ab . Then, x' is a vertex of the polytope P' , as otherwise x is not a vertex of $Q(G)$. Let G' be the graph with $V(G') = V$ and $E(G') = E \setminus \{e\}$. Then

$$P' = P(G') \cap \{x \in \mathbb{R}^{E(G')} : x(\delta^{>a}(b)) + x(\delta^{>b}(a)) \geq 1\},$$

since $P(G') = Q(G')$ by our minimality assumption. Define H' to be the hyperplane $\{x \in \mathbb{R}^{E(G')} : x(\delta^{>a}(b)) + x(\delta^{>b}(a)) = 1\}$. Then every vertex of P' is either a vertex of $P(G')$ or the intersection of an edge of $P(G')$ with the hyperplane H' . Since the vertices of $P(G')$ are integral, it remains to consider vertices of P' at the intersection of H' and an edge of $P(G')$. Such an edge would be defined by distinct stable matchings M_1 and M_2 , where the vertex of P' under consideration is not $\chi(M_1)$ nor $\chi(M_2)$. Note, that none of $\chi(M_1), \chi(M_2)$ lies on the hyperplane H' , since x' is the unique common point of H' and the line segment between $\chi(M_1)$ and $\chi(M_2)$. Thus, $|M_1 \cap (\delta^{>a}(b) \cup \delta^{>b}(a))| \neq 1$ and $|M_2 \cap (\delta^{>a}(b) \cup \delta^{>b}(a))| \neq 1$. On the other hand, the line segment between $\chi(M_1)$ and $\chi(M_2)$ has a nonempty intersection with H' , so w.l.o.g. we may assume that $|M_1 \cap (\delta^{>a}(b) \cup \delta^{>b}(a))| = 2$ and $|M_2 \cap (\delta^{>a}(b) \cup \delta^{>b}(a))| = 0$. Therefore, M_1 and M_2 satisfy (3) for the given edge ab . So Corollary 2.5 readily implies that $P(G')$ cannot have an edge connecting M_1 and M_2 .

Case $x_{ab} = 1$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$. Let x' be obtained by dropping the coordinates corresponding to $\delta(a) \cup \delta(b)$, and let G' be the graph with $V(G') = V \setminus \{a, b\}$ and $E(G') = E \setminus (\delta(a) \cup \delta(b))$. It is straightforward to see that x' is a vertex of $Q(G')$. Due to minimality assumption $P(G') = Q(G')$ and thus both x' and x are integral, a contradiction. ■

References

- [GS62] David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.
- [Rat96] Guillaume Ratier. On the stable marriage polytope. *Discrete Mathematics*, 148(1):141–159, 1996.
- [Rot92] Uriel G Rothblum. Characterization of stable matchings as extreme points of a polytope. *Mathematical Programming*, 54(1-3):57–67, 1992.
- [RS92] Alvin E Roth and Marilda A Oliveira Sotomayor. *Two-sided matching: A study in game-theoretic modeling and analysis*, chapter 2, page 37. Cambridge University Press, 1992.
- [Vat89] John H Vande Vate. Linear programming brings marital bliss. *Operations Research Letters*, 8(3):147–153, 1989.